Inverse Bremsstrahlung with High-Intensity Radiation Fields

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The absorption of high-intensity radiation, by the inverse bremsstrahlung process, has been studied. The results are significantly modified, compared to the results of perturbation theory, when the parameter $e^2E^2/m\hbar\omega^3$ becomes comparable to or greater than unity. Here E is the strength of the radiation electric field of frequency ω. It is found that the absorption cross section changes from an inverse seven-halves-power dependence on frequency for small values of the parameter, to direct proportionality for large values. Furthermore, for large radiation fluxes, the cross section varies inversely as the three-halves power of the flux.

I. INTRODUCTION

INVERSE bremsstrahlung refers to the process in which an electron absorbs radiation as it scatters in the Coulomb field of an ion. Theoretical studies of the bremsstrahlung process have included the assumption that the interaction of the electron with the radiation field may be treated by lowest order perturbation theory.[1] In the very early treatments it was also assumed that the electron is sufficiently energetic, both before and after absorption of radiation, that the electron-ion scatter may also be described by lowest-order perturbation theory. Subsequent efforts have permitted this latter condition to be relaxed by including the use of more exact Coulomb wave functions.[2,3]

The introduction of lasers[4] into current technology has produced an interest in the nonlinear interaction of radiation with electrons. A considerable effort has been expended in predicting corrections to the Thompson formula for the scattering of radiation from a free electron, by including multiple photon transfer processes.[5–7] The corrections to previous results have been found to be extremely small, even for the most intense radiation fields available; the expansion parameter for the free-electron scattering process may be written in the form[5–7]

$$(eE/m\omega c)^2\,,$$

where E is the electric field intensity for the radiation field of frequency ω, e, and m are the electron charge and mass, and c is the velocity of light. It is clear that the strong field correction is essentially a relativistic effect.

Recently, von Roos and others[8–10] have been quite successful in extending to atomic systems, techniques used in plasma studies. In these works, the statistical nature of the atom is emphasized, so that the starting point has been the Thomas-Fermi model, and use is made of a quantum-mechanical analog of the Vlasov equation.[11] Quantum correction to the Thomas-Fermi model, due to both exchange effects and strong potential gradients, are obtained in this manner. In our work, we will extend certain techniques used in plasma studies to the problem of a single electron which interacts with radiation.

It is of interest to consider the strong-field parameter associated with the inverse bremsstrahlung process, in order to determine the magnitude of the correction which we will consider in this work. This may be done by observing, from a classical point of view, how the nonlinearity arises. In the presence of the radiation field, an electron undergoes oscillation, with peak velocity

$$u_0 = eE_0/m\omega\,,$$

where E_0 is the peak value of the electric field strength. If $\mathbf{u}_0$ is comparable to the initial electron velocity $\mathbf{v}_0$, then this initial velocity loses its previous significance. Of greater interest is what happens when $\frac{1}{2}mu_0^2$ exceeds $\hbar\omega$. Then regardless of the initial electron velocity, the electron acquires sufficient energy, by interaction with the radiation field, such that it may emit photons. This is clearly a nonlinear effect, since the field has given the electron the energy which allows it to modify the field. The parameter which we seek, therefore, is

$$mu_0^2/2\hbar\omega \approx e^2E_0^2/m\hbar\omega^3\,.$$

From the quantum-mechanical point of view, a non-negligible magnitude of this parameter assures the importance of multiple photon transfer. For a 10^{12} W/cm^2 laser beam of infrared, with an angular frequency of $\omega = 10^{15}$ sec^{-1}, the magnitude of the parameter is of the order of unity. In this work, we consider how the absorption process is modified with these strong fields.

It will be assumed that relativistic effects are unimportant, that is, the initial electron energy $\frac{1}{2}mv_0^2$, the photon energy $\hbar\omega$, and the energy acquired by interaction with the field, $\frac{1}{2}mu_0^2$, are all assumed to be small compared to the electron rest energy. Furthermore, classical theory will be used whenever it is applicable. For example, since electron recoil velocities are nonrelativistic, the radiation field may be treated classically without introducing a further approximation. Further-

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more, the ion, which provides the background field in which absorption may occur, is much heavier than the electron. Therefore, ion recoil may be neglected, and the ion motion may therefore be treated classically; the resulting error is of the order of the electron-ion mass ratio. The electron motion is of course treated quantum mechanically. Finally, the electron-ion scatter will be described by the Born approximation. The use of free-state wave functions in lieu of Coulomb wave functions renders the mathematics considerably more tractable. Thus the electron velocity is, at all times, required to exceed greatly the quantity $eq/\hbar$, where e and q are the electron and ion charges, respectively.

We now describe the method of solution. In the center-of-mass coordinate system, the ion is effectively stationary, whereas the electron motion is extremely complicated. For example, the electron velocity is composed of three parts: the initial velocity; the increment attained by scattering in the ion field; and the oscillating component which results from interaction with the radiation field. In order to reduce the complexity of the electron motion, we transform to an oscillating coordinate system, so that it appears as if the ion is oscillating while the electron is unaffected by the radiation field. Now the ion produces a time-dependent field in which the electron scatters; in that sense, the simplification is somewhat illusory. We will refer to this latter coordinate system as the oscillating system, to distinguish it from the center of mass system.

In the oscillating system, the electron wave function, in the presence of the ion field, is time-dependent. Since the ion motion is classical, we may use the classical concept of force when referring to the ion. We conjecture that, the electron wave function being time-dependent, it transmits a reaction force to the ion. From the usual point of view, this force on the ion is merely due to an electron-ion scatter. But we can also consider this force from a somewhat different point of view.

Let us interpret the electron wave function ψ, such that $e\psi^*\psi$ represents an extended distribution of charge, which can support the propagation of longitudinal waves. The ion, oscillating in this medium, is a source of such waves. Since longitudinal waves carry energy; there is a reaction force associated with the emission of these waves by the ion. It is asserted that the force on an ion, calculated in this manner, is identical with the force associated with a particle-particle scatter.

We will denote the reaction force on the ion by $\mathbf{F}$. In the oscillating coordinate system, the time rate of energy which the ion absorbs from the radiation field is $\mathbf{F}\cdot\mathbf{u}$, where $\mathbf{u}$ is the ion velocity. We are really interested in the time-rate of energy absorption by the electron in the center-of-mass system. If the electron were a classical particle, then since Coulomb forces obey the law of action and reaction, the force which the ion exerts on the electron must be identical with the force which the electron exerts on the ion $\mathbf{F}$. The semiquantum statement of Newton's third law is that the time rate of momentum transfer from the ion to the electron is equal to $\mathbf{F}$. Similarly, the quantity $\mathbf{F}\cdot\mathbf{u}$ is the time rate of energy transfer from the ion to the electron.

Now the only reason that the ion transfers energy to the electron is that the ion is caused to oscillate by the radiation field. A static Coulomb field can result in no energy transfer. We conclude, therefore, that $\mathbf{F}\cdot\mathbf{u}$ is the energy which the radiation field transmits to the electron, by using the ion as an intermediary. But this quantity is what we seek; dividing it by the incident radiation flux per unit area yields the absorption cross section.

By using the technique described above, it is possible to treat the interaction of radiation with the electron to all orders in the field strength. In the next section, a general expression is obtained for the energy transfer. In the third section, this expression is reduced in the limits of weak and strong fields, respectively. It is shown that the extreme weak field limit is identical with the results of perturbation theory.

II. ENERGY TRANSFER

Let $\psi(t,\mathbf{r})$ be the electron wave function, so that $e\psi^*\psi$ represents its charge distribution. Similarly, let $qS(t,\mathbf{r})$ be the ion charge distribution. The potential field, $\varphi(t,\mathbf{r})$, produced by both particles, is obtained from Poisson's equation,

$$\nabla^2\varphi = 4\pi e\psi^*\psi - 4\pi qS(t,\mathbf{r}). \tag{1}$$

We now consider the quantities to be substituted into the right-hand side of this equation.

The electron wave function is obtained from the Schrödinger equation,

$$i\hbar\dot{\psi} = H_0\psi - e\varphi\psi; \quad H_0 = p^2/2m, \tag{2}$$

where the potential φ of Eq. (2) is the solution of Eq. (1). Note that the electron contributes to the potential field which acts back on the electron. Aside from resulting in self-energy effects, for an extended electron distribution, this procedure insures the collective effects needed for the propagation of longitudinal waves.

In order to solve Eq. (2), $\varphi(t,\mathbf{r})$ is treated as a perturbation. We have already agreed to treat the static Coulomb field by perturbation theory. But φ represents a time-dependent Coulomb field, which is modified by the presence of an electron distribution. Comparing φ with the static Coulomb field, it is clear that the presence of the electron can only weaken the perturbation, since the sign of its charge is opposite to that of the ion. Furthermore, the time dependence, produced solely by the ion motion, cannot induce a sizeable perturbation, except during those rare periods when the ion velocity is very nearly equal to the electron velocity.

The solution of Eq. (2) may be obtained in the Born approximation, in terms of the field φ, without specifying the field. The perturbation in the electron distribution,

$$\rho'(t,\mathbf{r})\equiv\psi_0{}^*\psi'+\psi_0\psi'^*, \qquad (3)$$

may then be calculated. The unperturbed part of the distribution, $\rho_0\equiv\psi_0{}^*\psi_0$, is independent of time, and is therefore unrelated to our problem.

When the solution of Eq. (2) is substituted into the time-space Fourier transform of Eq. (3), we obtain

$$\rho'(\Omega,\mathbf{k})=(e\varphi(\Omega,\mathbf{k})/L^3)\{[E(\mathbf{p}_0+\hbar\mathbf{k})-E(\mathbf{p}_0)-\hbar\Omega]^{-1}+[E(\mathbf{p}_0-\hbar\mathbf{k})-E(\mathbf{p}_0)+\hbar\Omega]^{-1}\}, \qquad (4)$$

where

$$\rho'(\Omega,\mathbf{k})\equiv\int\rho'(t,\mathbf{r})\exp(i\Omega t-i\mathbf{k}\cdot\mathbf{r})dtd^3r \qquad (5)$$

is the transform of the perturbation in the electron distribution, $\varphi(\Omega,\mathbf{k})$ is the corresponding transform of the field $\varphi(t,\mathbf{r})$, L^3 is the normalization volume for the electron of initial momentum $\mathbf{p}_0$, and

$$E(\mathbf{p}_0)=\mathbf{p}_0{}^2/2m \qquad (6)$$

is the electron energy. When expression (4) is substituted into the time-space Fourier transform of Poisson's equation (1), we obtain as the solution for the field,

$$\varphi(\Omega,\mathbf{k})=4\pi q[S(\Omega,\mathbf{k})/D(\Omega,\mathbf{k})], \qquad (7)$$

where

$$D(\Omega,\mathbf{k})\equiv k^2+(4\pi e^2/L^3)\{[E(\mathbf{p}_0+\hbar\mathbf{k})-E(\mathbf{p}_0)-\hbar\Omega]^{-1}+[E(\mathbf{p}_0-\hbar\mathbf{k})-E(p_0)+\hbar\Omega]^{-1}\}, \qquad (8)$$

and $S(\Omega,\mathbf{k})$ is the transform of the ion source term, $S(t,\mathbf{r})$. We now consider this term.

The ion wave function Ψ must satisfy its own Schrödinger equation. To lowest order in the electron-ion interaction, we may neglect electron reaction effects. The corresponding Schrödinger equation for an ion in the radiation field is

$$i\hbar\dot{\Psi}=\frac{1}{2M}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2\Psi, \qquad (9)$$

where M is the ion mass, and $\mathbf{A}$ is the radiation vector potential. Since electrons (in the center-of-mass system) are not accelerated to relativistic velocities, we may treat the radiation field in the electric dipole limit. For example, the radiation vector potential may be written, quite generally, in the form

$$\mathbf{A}=(Mc\omega/e)\mathbf{r}_0\sin(\omega t-\mathbf{K}\cdot\mathbf{x}),$$

where $\mathbf{r}_0$ is defined, for convenience, such that $(Mc\omega/e)\mathbf{r}_0$ is the amplitude of the vector potential, and $\mathbf{K}$ is the radiation wave number (to be distinguished from the longitudinal field wave number $\mathbf{k}$, to be introduced later). The dipole approximation in the radiation field is valid if $Kx\ll1$, where x describes the region of the charge distribution. Since r_0 is the maximum displacement of the charge from its equilibrium position, the condition is

$$Kr_0=\frac{\omega}{c}\frac{eE_0}{m\omega^2}=\frac{u_0}{c}\ll1,$$

in accordance with our assumption that electrons are not accelerated to relativistic velocities by the radiation field. Corrections of order u_0/c are much smaller than the corrections to the Born approximation which are obtained in this work. It should be pointed out that multipole transitions are important in determining the corrections to Thomson scattering since all corrections to that process are relativistic, of the order $(u_0/c)^2$. In our approximation, however, the vector potential depends only on time, and is given by

$$\mathbf{A}=(Mc\omega/e)\mathbf{r}_0\sin\omega t. \qquad (10)$$

The solution of Eqs. (9) and (10), for the wave function corresponding to an ion with initial momentum $\mathbf{p}_1$, is

$$\Psi_1=L^{-3/2}\exp\left\{i\frac{\mathbf{p}_1}{\hbar}\cdot\mathbf{r}-i\frac{E_s(\mathbf{p}_1)}{\hbar}t-i\frac{\mathbf{p}_1\cdot\mathbf{r}_0}{\hbar}\cos\omega t-i\frac{M\omega r_0{}^2}{4\hbar}[\omega t-\tfrac{1}{2}\sin(2\omega t)]\right\}, \qquad (11)$$

where L^3 is the normalization volume and

$$E_s(\mathbf{p}_1)=\mathbf{p}_1{}^2/2M \qquad (12)$$

is the ion energy. The quantity which corresponds to the classical density function, for a particle in quantum state Ψ_1, is given by

$$S(t,\mathbf{r})=\sum_m\Psi_m{}^*\Psi_1=\frac{L^3}{(2\pi\hbar)^3}\int\Psi_m{}^*\Psi_1d^3p_m, \qquad (13)$$

where the summation is replaced by an integration over all momentum eigenstates of the system. By substituting expression (11) into (13) we find that the Fourier transform of Eq. (13) may be written

$$S(\Omega,\mathbf{k})=\int_{-\infty}^{\infty}\exp\left\{\frac{i}{\hbar}[E_s(\mathbf{p}_1-\hbar\mathbf{k})-E_s(\mathbf{p}_1)+\hbar\Omega]t-i\mathbf{k}\cdot\mathbf{r}_0(t)\right\}dt$$

$$=\int_{-\infty}^{\infty}\exp\left\{i\left(\Omega-\mathbf{k}\cdot\mathbf{V}+\frac{\hbar k^2}{2M}\right)t-i\mathbf{k}\cdot\mathbf{r}_0(t)\right\}dt, \qquad (14)$$

where $\mathbf{V}$ is the initial velocity of the ion and $\mathbf{r}_0(t)=\mathbf{r}_0\cos\omega t$. This last expression will often be written as $\mathbf{r}_0(t)$, since many of the results are valid when the radiation field has a distribution of frequency components.

In order to obtain the classical distribution, we drop the term $(\hbar k^2/2M)t$ from Eq. (14). Then spreading of the ion wave packet is ignored. When typical wave numbers $\mathbf{k}$, corresponding to emitted longitudinal waves are obtained, it may be shown that the resulting error is of the order of the electron-ion mass ratio. Furthermore, we will assume that the ion is initially at rest, and set $\mathbf{V}=0$. Then Eq. (14) is reduced to

$$S(\Omega,\mathbf{k})=\int_{-\infty}^{\infty}\exp\{i\Omega t-i\mathbf{k}\cdot\mathbf{r}_0(t)\}dt \tag{15}$$

which is to be combined with Eq. (7).

The electric potential $\varphi(\Omega,\mathbf{k})$ describes the amplitude of longitudinal waves emitted by the ion. According to Eq. (15), $\mathbf{r}_0(t)$ defines the position of the classical ion at all times. Therefore, the reaction force on the ion, due to the emission of these waves, is given by

$$\mathbf{F}(t)=-q\nabla\varphi(t,\mathbf{r}_0(t))=-\frac{iq}{(2\pi)^4}\int\mathbf{k}\varphi(\Omega,\mathbf{k})\times\exp\{-i\Omega t+i\mathbf{k}\cdot\mathbf{r}_0(t)\}d\Omega d^3k. \tag{16}$$

By substituting Eqs. (7) and (15) into (16), we obtain

$$\mathbf{F}(t)=-i\frac{q^2}{4\pi^3}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}d\Omega\times\int\frac{\mathbf{k}\exp\{i\Omega(t'-t)-i\mathbf{k}\cdot[\mathbf{r}_0(t')-\mathbf{r}_0(t)]\}}{D(\Omega,\mathbf{k})}d^3k, \tag{17}$$

where $D(\Omega,\mathbf{k})$ is given by Eq. (8). To obtain the force, the real part of Eq. (17) must be taken.

For a sinusoidal radiation field, we have $\mathbf{r}_0(t)=\mathbf{r}_0\cos\omega t$. Thus factors of Eq. (17), of the form $\exp\{i\mathbf{k}\cdot\mathbf{r}_0\cos\omega t\}$, may be expanded in terms of Bessel functions, $J_n(\mathbf{k}\cdot\mathbf{r}_0)$. We then find

$$\mathbf{F}(t)=-i\frac{q^2}{4\pi^2}\sum_{m,n=-\infty}^{\infty}(-1)^n i^{n+m}\times\int\frac{\mathbf{k}J_n(\mathbf{k}\cdot\mathbf{r}_0)J_m(\mathbf{k}\cdot\mathbf{r}_0)}{|D(n\omega,\mathbf{k})|^2}\cdot\{D^*(n\omega,\mathbf{k})e^{i(m-n)\omega t}-(-1)^{n+m}D(n\omega,\mathbf{k})e^{-i(m-n)\omega t}\}d^3k, \tag{18}$$

where the real part of the function has been taken. A complete knowledge of the force, along with its time dependence, would enable us to obtain the radiation phase shift on scattering, as well as the absorption. An inspection of Eq. (18), however, indicates serious mathematical difficulties. We will, therefore, limit our study to absorption. Furthermore, for radiation frequencies of interest to this work, no measurements can be made in less time than a period of the wave. We need, therefore, consider only the quantity

$$\dot{U}=\langle\mathbf{F}\cdot\mathbf{u}\rangle, \tag{19}$$

where

$$\mathbf{u}(t)=d\mathbf{r}_0(t)/dt=-\omega\mathbf{r}_0\sin\omega t \tag{20}$$

is the ion velocity, and the brackets indicate that we take a time average over a period of the wave. By combining Eqs. (18) and (20) with (19), we get

$$\dot{U}=-\frac{q^2\omega}{\pi^2}\operatorname{Im}\sum_{n=1}^{\infty}n\int\frac{J_n^2(\mathbf{k}\cdot\mathbf{r}_0)D(n\omega,\mathbf{k})}{|D(n\omega,\mathbf{k})|^2}d^3k. \tag{21}$$

Since the electron is distributed over a large volume, the term k^2 in Eq. (8) for $D(\Omega,\mathbf{k})$ is by far the largest term. Therefore, to lowest order, we may replace $D(n\omega,\mathbf{k})$, in the denominator of Eq. (21), by k^2. The result is

$$\dot{U}=-\frac{q^2\omega}{\pi^2}\operatorname{Im}\sum_{n=1}^{\infty}n\int J_n^2(\mathbf{k}\cdot\mathbf{r}_0)D(n\omega,\mathbf{k})\frac{d^3k}{k^4}. \tag{22}$$

By inspection of Eq. (8), we find that $\operatorname{Im}D(\Omega,\mathbf{k})$ is still not completely defined. Since $\operatorname{Im}(1/x)=\pm\pi\delta(x)$, it is still necessary to prescribe the contour around the poles in order to obtain the proper signs. From causality arguments, we assert that the poles are defined such that

$$\operatorname{Im}D(\Omega,\mathbf{k})=\frac{4\pi^2e^2}{L^3}\{\delta(E(\mathbf{p}_0+\hbar\mathbf{k})-E(\mathbf{p}_0)-\hbar\Omega)-\delta(E(\mathbf{p}_0-\hbar\mathbf{k})-E(\mathbf{p}_0)+\hbar\Omega)\}. \tag{23}$$

The quantity $\operatorname{Re}D(\Omega,\mathbf{k})$ plays a part in phase shift studies, but is not involved in the absorption process.

In order to simplify the mathematics, we will assume that the radiation propagates in the direction of the initial electron momentum, $\mathbf{p}_0=m\mathbf{v}_0$. Then by symmetry the result must be independent of the direction of polarization, $\hat{r}_0$. For this problem we find by substituting expression (23) into (22), and performing one angular integral over the δ functions,

$$\dot{U}=-\frac{8e^2q^2\omega}{L^3\hbar v_0}\sum_{n=1}^{\infty}n\int\frac{dk}{k^3}\int J_n^2(kr_0\mu)\times\left\{\left[1-\left(\frac{n\omega}{kv_0}-\frac{\hbar k}{2mv_0}\right)^2-\mu^2\right]^{-1/2}-\left[1-\left(\frac{n\omega}{kv_0}+\frac{\hbar k}{2mv_0}\right)^2-\mu^2\right]^{-1/2}\right\}d\mu, \tag{24}$$

where the limits of integrations include all regions where the integrand is defined. For example, when integrating over the first term in the curly brackets, the integration limits on k include values of k such that

$$\left(\frac{n\omega}{kv_0}-\frac{\hbar k}{2mv_0}\right)^2\leq 1.$$

The corresponding limits are

$$k_1=\frac{mv_0}{\hbar}\left[\left(1+\frac{2n\hbar\omega}{mv_0^2}\right)^{1/2}\pm1\right]. \tag{25}$$

The limits for the second term in the curly brackets are

$$k_2=\frac{mv_0}{\hbar}\left[1\pm\left(1-\frac{2n\hbar\omega}{mv_0^2}\right)^{1/2}\right]. \tag{26}$$

For terms in the sum, such that $n\hbar\omega>\frac{1}{2}mv_0^2$, the second term of Eq. (24) does not appear. The limits on the μ integrations are

$$\binom{\mu_1}{\mu_2}=\frac{\pm}{\pm}\left[1-\left(\frac{n\omega}{kv_0}\mp\frac{\hbar k_0}{2mv_0}\right)^2\right]^{1/2}. \tag{27}$$

The first term in the curly brackets of Eq. (24) is interpreted as describing, for a given n, the net absorption over induced emission, of n photons from the radiation field. The second term corresponds to the net induced emission over absorption of n photons.

The absorption cross section for inverse bremsstrahlung is related to $\dot{U}$, as given by Eq. (24), by

$$\sigma=-\frac{8\pi}{cE_0^2}L^3\rho_i\dot{U}=-\frac{8\pi e^2\rho_iL^3}{cm^2\omega^4r_0^2}\dot{U}, \tag{28}$$

where $cE_0^2/8\pi$ is the incident energy flux. The factor $L^3\rho_i$ must be included in order to undo the arbitrariness of having normalized the electron wave function in the volume L^3. The ion density is used rather than the electron density because, in the center-of-mass coordinate system, the electron is the source particle for longitudinal waves, rather than the ion. For the same reason, we have written the ion displacement in the radiation field as

$$\mathbf{r}_0=-e\mathbf{E}_0/m\omega^2, \tag{29}$$

where m is the electron mass.

III. LIMITING CASES

By observing the limits of integration of Eq. (24), as given by Eqs. (25) and (26), we see that there is a distribution of wave numbers associated with the longitudinal waves. We will first assume that for all such values of k, we have

$$kr_0\ll 1. \tag{30}$$

Then we need retain only the $n=1$ term in the series of Eq. (24). Furthermore, the Bessel function may be reduced to its limiting value

$$J_1(kr_0\mu)\approx\tfrac{1}{2}kr_0\mu.$$

In the weak field limit described here, the integrations of Eq. (24) may be performed immediately. By combining the result with Eq. (28), we have for the cross section

$$\sigma=\frac{8\pi^2e^4q^2\rho_i}{\hbar cm^2\omega^3v_0}\left\{\left[\left(1+\frac{x}{2}\right)\ln\left(\frac{(1+x)^{1/2}+1}{(1+x)^{1/2}-1}\right)-(1+x)^{1/2}\right]\right.$$
$$\left.-\left[\left(1-\frac{x}{2}\right)\ln\left(\frac{1+(1-x)^{1/2}}{1-(1-x)^{1/2}}\right)-(1-x)^{1/2}\right]\right\},\quad x<1, \tag{31}$$

where

$$x\equiv 2\hbar\omega/mv_0^2. \tag{32}$$

For $x>1$, the second term of Eq. (31) is discarded. In this extreme limit, since only a single photon is involved, the processes of absorption and induced emission are separable. For any higher order terms, this is no longer the case. Equation (31) is identical with the result obtained from the perturbation treatment of the nonrelativistic inverse bremsstrahlung process, when the radiation propagates in the same direction as the electron velocity.[12] It is not difficult to show that, for $kr_0\ll 1$, Eqs. (22) and (23) agree with previously quoted results for arbitrary directions of propagation.

If we include one higher order in r_0^2, so that two quantum processes are possible, we obtain as the correction to expression (31)

$$\sigma=\sigma_0+\sigma',$$

where

$$\sigma'=\frac{8\pi^2e^4q^2\rho_i}{\hbar cm^2\omega^3v_0}\left(\frac{mv_0r_0}{\hbar}\right)^2\left\{\frac{3}{32}\left[x^2\left(1+\frac{x}{2}\right)\ln\left(\frac{(1+x)^{1/2}+1}{(1+x)^{1/2}-1}\right)\right.\right.$$
$$\left.-x^2(1+x)^{1/2}-\frac{4}{3}(1+x)^{3/2}+\frac{4}{3}\right]$$
$$-\frac{3}{16}\left[x^2(1+x)\ln\left(\frac{(1+2x)^{1/2}+1}{(1+2x)^{1/2}-1}\right)\right.$$
$$\left.\left.-x^2(1+2x)^{1/2}-\frac{1}{3}(1+2x)^{3/2}+\frac{1}{3}\right]\right\}, \tag{31'}$$

for $x>1$. For $\frac{1}{2}<x<1$, a term identical with the first term in the brackets, but with x replaced by $-x$, is subtracted from this expression. For $x<\frac{1}{2}$, both brackets, with x replaced by $-x$, are subtracted.

We now consider the alternate to condition (30), that is,

$$kr_0\gg 1. \tag{33}$$

Rather than attempt to evaluate expression (24), which may involve many terms in the sum, we return to Eq. (17). According to Eq. (33), we see that the integrand of Eq. (17) is rapidly oscillating, except over

[12] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1954), see Chap. V, p. 242. (In order to obtain precise agreement, Heitler's study must be modified slightly to include the inverse process.)

a very small distribution in angles of $\mathbf{k}$, which may be neglected, and except in the region $t'\approx t$. Therefore, we perform a Taylor expansion,

$$\mathbf{r}_0(t')-\mathbf{r}_0(t)=\mathbf{u}(t)(t'-t)+\frac{1}{2}\frac{d\mathbf{u}}{dt}(t'-t)^2+\frac{1}{6}\frac{d^2\mathbf{u}}{dt^2}(t-t)^3+\cdots. \quad (34)$$

We need retain only the first two terms in this expansion if

$$\mathbf{k}\cdot(d^2\mathbf{u}/dt^2)(t'-t)^3\ll 1$$

when

$$\mathbf{k}\cdot(d\mathbf{u}/dt)(t'-t)^2\approx 1.$$

Except for that small range in angles, referred to above, we find that condition (33) ensures that the first two terms in the expansion of Eq. (34) are sufficient. Therefore, Eq. (17) is reduced to

$$\mathbf{F}(t)=-i\frac{q^2}{4\pi^3}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}d\Omega \times\int\frac{\mathbf{k}\exp\{i(\Omega-\mathbf{k}\cdot\mathbf{u})(t'-t)-\frac{1}{2}i\mathbf{k}\cdot(d\mathbf{u}/dt)(t'-t)^2\}}{D(\Omega,\mathbf{k})}d^3k. \quad (35)$$

After performing the integration over t', and changing the variable of integration from Ω to x, by

$$\Omega=\mathbf{k}\cdot\mathbf{u}+(\mathbf{k}\cdot d\mathbf{u}/dt)^{1/2}x,$$

we have

$$F(t)=-i\frac{q^2}{4\pi^3}(2\pi i)^{1/2}\int_{-\infty}^{\infty}e^{ix^2/2}dx \times\int\frac{\mathbf{k}d^3k}{D[\mathbf{k}\cdot\mathbf{u}+(\mathbf{k}\cdot(d\mathbf{u}/dt))^{1/2}x,\mathbf{k}]}. \quad (36)$$

Recalling again, that according to Eq. (8), k^2 is the dominant contribution to D, Eq. (36) may be replaced by

$$F(t)=i\frac{q^2}{4\pi^3}(2\pi i)^{1/2}\int_{-\infty}^{\infty}e^{ix^2/2}dx \times\int\mathbf{k}D\left[\mathbf{k}\cdot\mathbf{u}+\left(\mathbf{k}\cdot\frac{d\mathbf{u}}{dt}\right)^{1/2}x,\mathbf{k}\right]\frac{d^3k}{k^4}. \quad (37)$$

To lowest order in $(\mathbf{k}\cdot\mathbf{r}_0)^{-1}$, we may neglect the acceleration term, $(\mathbf{k}\cdot d\mathbf{u}/dt)^{1/2}x$, in the argument of D. Then we find by substituting Eq. (23) into Eq. (37), that

$$\mathbf{F}(t)=-\frac{2e^2q^2}{L^3}\int\mathbf{k}\left\{\delta\left(\hbar\mathbf{k}\cdot(\mathbf{u}-\mathbf{v}_0)-\frac{\hbar^2k^2}{2m}\right)-\delta\left(\hbar\mathbf{k}\cdot(\mathbf{u}-\mathbf{v}_0)+\frac{\hbar^2k^2}{2m}\right)\right\}\frac{d^3k}{k^4}. \quad (38)$$

After performing the integrations and combining the result with Eqs. (19) and (28), we have the result

$$\sigma=\frac{32\pi^2e^4q^2\rho_i}{cm^3\omega^4r_0^2}\left\langle\frac{u^2-\mathbf{u}\cdot\mathbf{v}_0}{|\mathbf{u}-\mathbf{v}_0|^3}\int^{k_0}\frac{dk}{k}\right\rangle, \quad (39)$$

where

$$k_0=(2m/\hbar)|\mathbf{u}-\mathbf{v}_0|. \quad (40)$$

The lower limit of integration of Eq. (39) is, according to this approximation, $k_{\min}=0$. Of course, by condition (33), this approximation becomes invalid for $k\lesssim 1/r_0$. Furthermore, retention of the acceleration term in Eq. (37) will provide a lower limit cutoff on k. For example, we have from Eq. (37), with $\mathbf{v}_0=0$ and $d\mathbf{u}/dt$ in the same direction as $\mathbf{u}$,

$$\mathbf{F}(t)=-i\frac{2e^2q^2}{\hbar L^3u}(2\pi i)^{1/2}\hat{u}\int_{-\infty}^{\infty}e^{ix^2/2}dx \times\int_0^{\infty}\frac{dk}{k^2}\int_{-1}^{1}\mu\left\{\delta\left(\mu-\frac{\hbar k}{2mu}+\left(\frac{ax^2}{ku^2}\right)^{1/2}|\mu|^{1/2}\right)-\delta\left(\mu+\frac{\hbar k}{2mu}+\left(\frac{ax^2}{ku^2}\right)^{1/2}|\mu|^{1/2}\right)\right\}d\mu, \quad (41)$$

where $a\equiv|d\mathbf{u}/dt|$ and $\hat{u}$ is a unit vector in the direction of $\mathbf{u}$. With condition (33), the upper limit on the integration over k, obtained by setting the argument of the δ function to vanish, with $\mu=\pm 1$, is virtually unchanged from the value $k_0=2mu/\hbar$. The value of the two inner integrals of Eq. (41) is, in limits,

$$\hbar k/2mu,$$

for

$$k^2\gg max^2/2\hbar u$$

and

$$(\hbar k/mu)(\hbar uk^2/2max^2),$$

for

$$k^2\ll max^2/2\hbar u.$$

Thus there is a fairly sharp cutoff at

$$k\approx(ma/2\hbar u)^{1/2}x,$$

and this value, with $x=1$, may be taken as $k_{\min}$ in the integral of Eq. (39). When $\mathbf{v}_0\neq 0$, there is a small logarithmic correction which results from modification in the cutoffs.

In order to allow a comparison between Eqs. (39) for $kr_0\gg 1$ and (31) for $kr_0\ll 1$, we set $\mathbf{u}\perp\mathbf{v}_0$ with the approximate result

$$\sigma=\frac{16\pi^2e^2q^2\rho_i}{cmE_0^2}\left\langle\frac{u^2}{(u^2+v_0^2)^{3/2}}\ln\left(\frac{8mu^3}{\hbar a}\right)\right\rangle. \quad (42)$$

The time dependence, to be averaged over, occurs primarily through the velocity,

$$u=(eE_0/m\omega)\sin\omega t.$$

A particularly interesting situation occurs for $\frac{1}{2}mv_0^2 \ll \hbar\omega$. Then we find that if $\frac{1}{2}mu^2 \ll \hbar\omega$, the cross section is given by the expression (31) and (31′), with $x \gg 1$. The result is

$$\sigma=\frac{32\pi^2e^4q^2\rho_i}{3\hbar cm^2\omega^3}(m/2\hbar\omega)^{1/2}\left\{1-\frac{9}{32}\left(1-\frac{1}{\sqrt{2}}\right)\frac{e^2E_0^2}{m\hbar\omega^3}+\cdots\right\},\quad \frac{e^2E_0^2}{m\hbar\omega^3}\ll 1. \tag{43}$$

For $\frac{1}{2}mu^2 \gg \hbar\omega$, we have by Eq. (42)

$$\sigma=\frac{16\pi e^4q^2\rho_i\omega}{c(eE_0)^3}\ln\left(\frac{eE_0}{m\omega v_0}\right)\ln\left(\frac{64mv_0^3eE_0}{\hbar^2\omega^3}\right),\quad \frac{e^2E_0^2}{m\hbar\omega^3}\gg 1. \tag{44}$$

It is predicted therefore, that with sufficiently strong radiation fluxes, and with the condition $\frac{1}{2}mv_0^2 \ll \hbar\omega$, the absorption cross section will decrease roughly as the three-halves power of the flux. As the intensity is increased, the frequency dependence goes from inverse seven-halves power to direct proportionality.

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Experiments on the Average Characteristics of Cascade Showers Produced in Lead by 500- and 1000-MeV Electrons*

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The average number and the energy spectrum of shower electrons present under various thicknesses of lead were obtained with the aid of multiplate and magnetic cloud chambers. The relation between observed track length and incident electron energy was found to be

incident energy (MeV) = (23.6±1.6)×track length (radiation lengths).

The observed number of shower electrons with energy greater than 10 MeV is in good agreement with that predicted by recent Monte Carlo calculations; however, low-energy electrons (not included in the calculations) were found to be a large fraction of those present at large depths. Measured probabilities p_n that exactly n electrons emerge from the lower surface of a 0.75-radiation-length lead plate when one electron is incident from above are given as a function of incident electron energy.

I. INTRODUCTION

The present experiment was originally undertaken to take advantage of the availability of artificially produced beams of energetic electrons for the limited purpose of "calibrating" balloon-borne cloud chambers that had been used to study cosmic-ray electrons. However, it soon became apparent that some of the results obtained were of sufficient general interest to warrant their presentation in this report.

Although existing experimental and theoretical studies have led to a clear understanding of the nature and major characteristics of cascade showers, detailed knowledge of showers developing in materials of high atomic number is needed because such showers provide a useful tool for determining the identity and energy of the initiating electron or photon. It is generally conceded that analytic shower theories[1] yield a useful and essentially correct representation of shower development in materials of low atomic number, but the analysis of showers in high-Z materials is complicated by the intractability of mathematical expressions for the low-energy cross sections of elementary shower processes and by uncertainties arising from the pronounced effects of multiple scattering on low-energy shower particles. These difficulties have been circumvented to a certain extent by Monte Carlo calculations based on exact expressions for the cross sections,[2–5] but even these calculations yield no information on the number of particles present with energy below an arbitrary low-energy cutoff which must be introduced to limit the extent of the computation. Unfortunately, published experimental data on showers in high-Z materials,[6–11] while extensive, are so disjointed that

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[1] Summaries of the results of shower theory appear in: B. Rossi, *High Energy Particles* (Prentice Hall, Inc., New York, 1952); and K. Greisen, *The Extensive Air Showers*, in *Progress in Cosmic Ray Physics* (North-Holland Publishing Company, Amsterdam, 1956), Vol. III, pp. 1–141.

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